Chapter 3

Continuous Distributions

3.1 Continuous-Type Data

3.1–2 $\overline{x} = 3.58$; $s = 0.5116$.

3.1–4 **(a)** The respective class frequencies are 2, 8, 15, 13, 5, 6, 1;

(b)

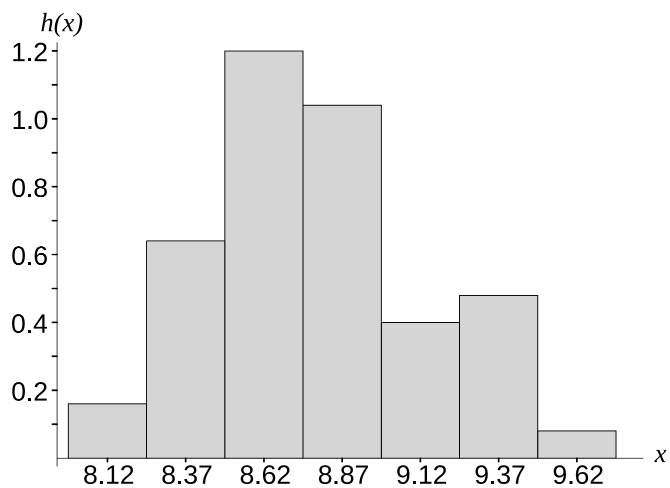


Figure 3.1–4: Weights of nails

(c) $\overline{x} = 8.773$, $\overline{u} = 8.785$, $s_x = 0.365, s_u = 0.352$;

(d) $800 * \overline{u} = 7028$, $800 * (\overline{u} + 2 * s_u) = 7591.2$. The answer depends on the cost of the nails as well as the time and distance required if too few nails are purchased.

3.1–6 **(a)**

Class Interval	Class Limits	Frequency f_i	Class Mark,u_i
(93.555, 101.555)	(93.56, 101.55)	5	97.555
(101.555, 109.555)	(101.56, 109.55)	11	105.555
(109.555, 117.555)	(109.56, 117.55)	22	113.555
(117.555, 125.555)	(117.56, 125.55)	26	121.555
(125.555, 133.555)	(125.56, 133.55)	22	129.555
(133.555, 141.555)	(133.56, 141.55)	22	137.555
(141.555, 149.555)	(141.56, 149.55)	8	145.555
(149.555, 157.555)	(149.56, 157.55)	4	153.555
(157.555, 165.555)	(157.56, 165.55)	3	161.555
(165.555, 173.555)	(165.56, 173.55)	2	169.555

(b)

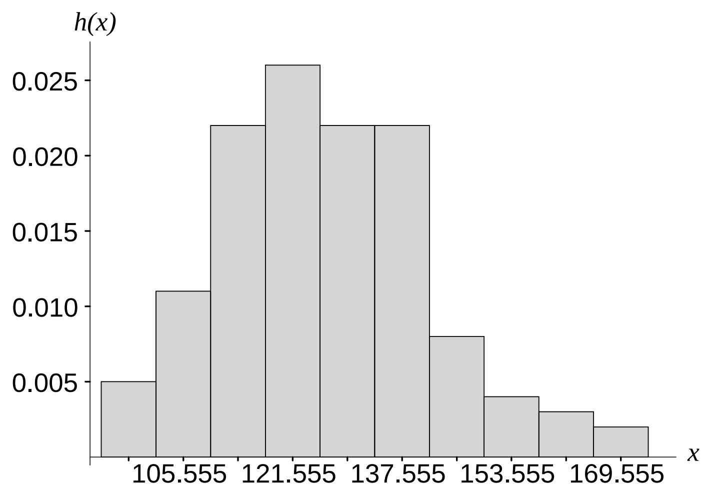


Figure 3.1–6: Old Kent River Bank Run times

(c) The histogram is skewed slightly to the right.

3.1–8 **(a)** With the class boundaries $3.5005, 3.5005, 3.6005, \dots, 4.1005$, the respective class frequencies are $4, 7, 24, 23, 7, 4, 3, 9, 15, 23, 18, 2$.

(b)

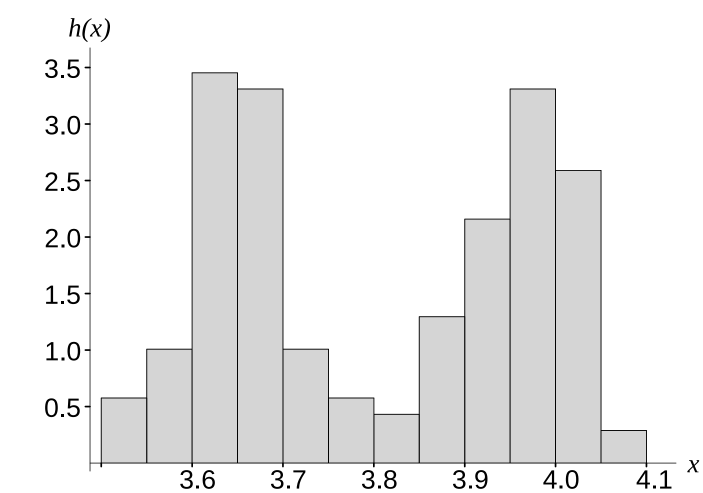


Figure 3.1–8: Weights of mirror parts

(c) This is a bimodal histogram.

3.1–10 **(a)** With the class boundaries $0.5, 5.5, 17.5, 38.5, 163.5, 549.5$, the respective frequencies are $11, 9, 10, 10, 10$.

(b)

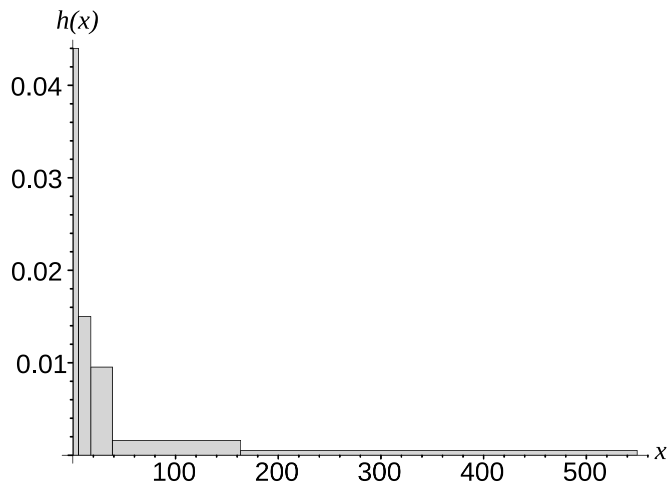


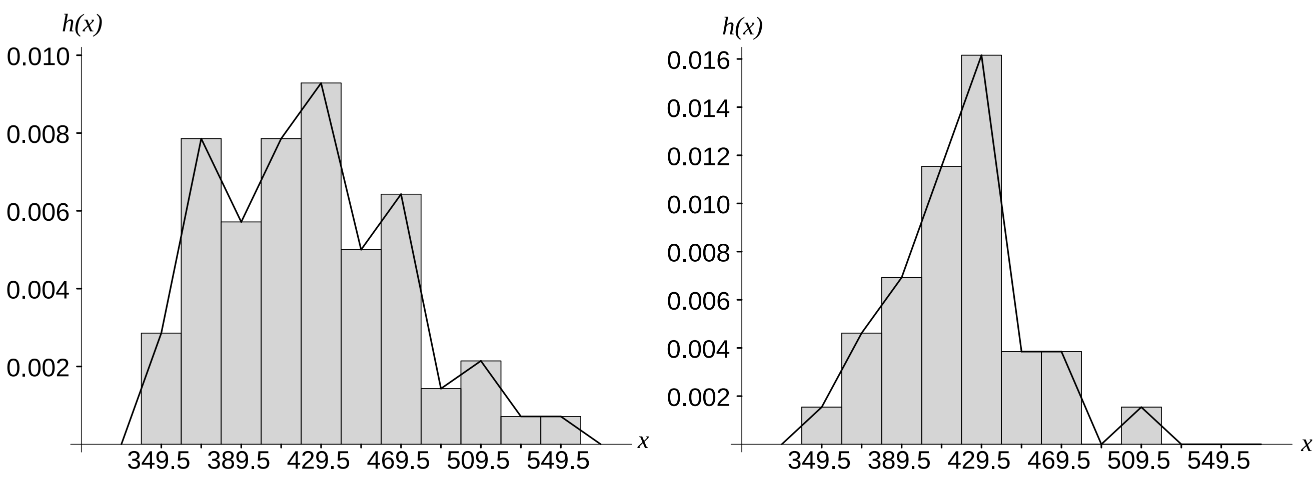
Figure 3.1–10: Mobil home losses

(c) This is a skewed to the right distribution.

3.1–12 **(a)**

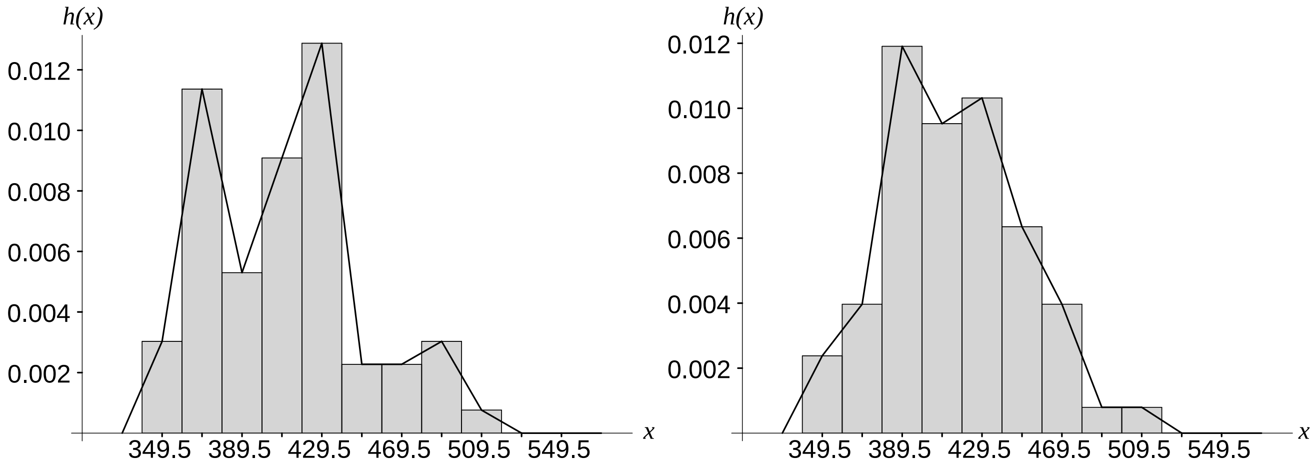
	Means		St. Devs.	
Player	1998	1999	1998	1999
McGwire	423.757	415.862	46.409	32.320
Sosa	407.485	412.016	38.136	33.197

(b)



Mark McGwire in 1998 Mark McGwire in 1999

Figure 3.1–12: Distances Mark McGwire's home runs traveled



Sammy Sosa in 1998 Sammy Sosa in 1999

Figure 3.1–12: Distances Sammy Sosa's home runs traveled

3.1–14 **(a)** $\overline{x} = 1.335$, $s^2 = 0.003971$, $s = 0.0630$;

(b)

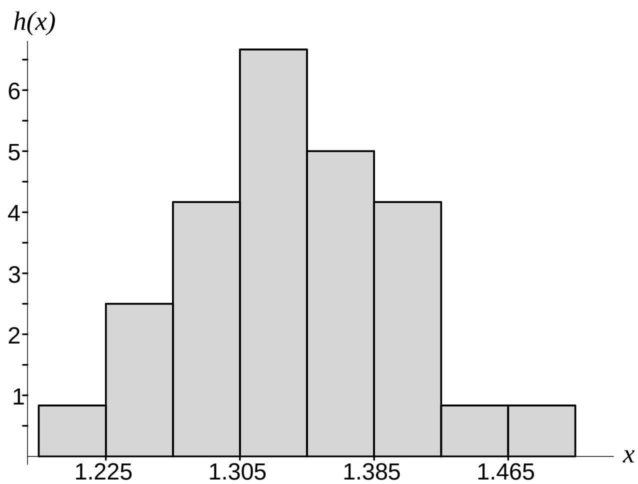


Figure 3.1–14: Diameters of grains of soil

3.1–16 **(a)**

Stems	Leaves	Frequency	Depths
20f	5	1	1
20s	6 6 7 7	4	5
20•	8 8 9 9 9	5	10
21∗	0 0 0 0 0 1 1	7	17
21t	2 2 2 2 2 3 3 3 3 3 3 3 3	13	30
21f	4 4 4 4 4 4 4 5 5 5 5 5 5 5 5	15	(15)
21s	6 6 6 6 6 6 6 6 6 6 6 6 6 7 7 7 7 7 7 7 7 7 7	23	36
21•	8 8 8 8 8 8 9 9 9 9	10	13
22∗	0 0 0	3	3

(Multiply numbers by 10^{-1}.)

(b) **(i)** $\widetilde{q}_1 = \frac{1}{2}(21.2 + 21.2) = 21.2$; $\widetilde{q}_2 = 21.5$; $\widetilde{q}_3 = \frac{1}{2}(21.7 + 21.7) = 21.7$;

(ii) $\widetilde{\pi}_{0.60} = (0.8)21.6 + (0.2)21.6 = 21.6;$

(iii) $\widetilde{\pi}_{0.15} = (0.7)21.0 + (0.3)21.0 = 21.0;$

3.1–18

Stems	Leaves	Frequency	Depths
0•	612	1	1
1∗	450	1	2
1•	560 889 961 994	4	6
2∗	065 142 151 172 195 290	6	12
2•	510 545 788 817 880 921 938	7	(7)
3∗	011 041 051 060 062 080 090	7	7

(Multiply numbers by 10^{-2}.)

3.2 Random Variables of the Continuous Type

3.2–2 **(a)** **(i)**
$$\begin{aligned} \int_0^c x^3/4\,dx &= 1 \\ c^4/16 &= 1 \\ c &= 2; \end{aligned}$$

(ii)
$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t)\,dt \\ &= \int_0^x t^3/4\,dt \\ &= x^4/16, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ x^4/16, & 0 \le x < 2, \\ 1, & 2 \le x < \infty. \end{cases}$$

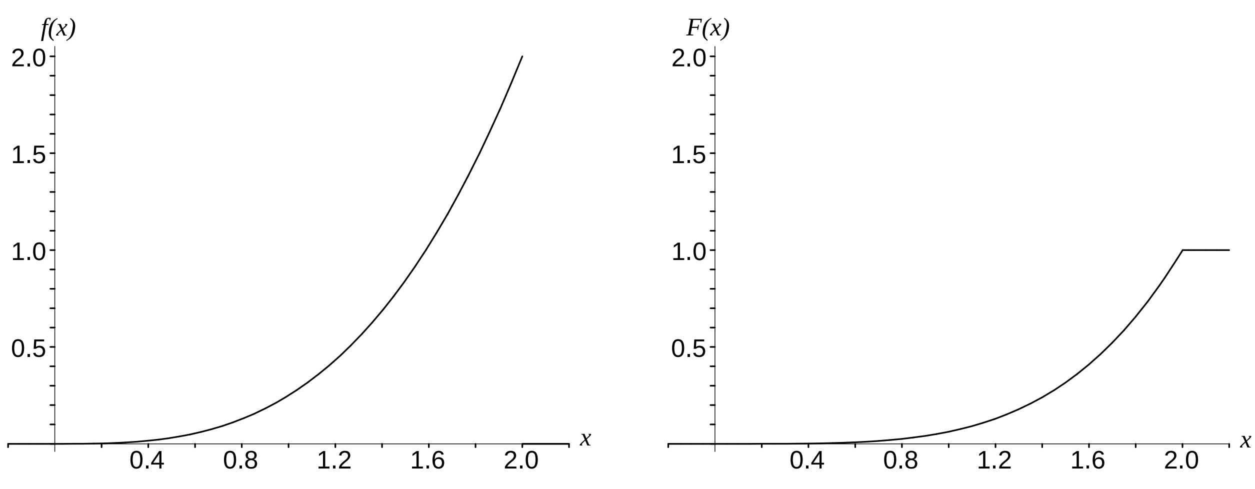


Figure 3.2–2: **(a)** Continuous distribution p.d.f. and c.d.f.

(b) **(i)**

$$\begin{aligned} \int_{-c}^{c} (3/16)x^2\,dx &= 1 \\ c^3/8 &= 1 \\ c &= 2; \end{aligned}$$

(ii)

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_{-2}^{x} (3/16)t^2\,dt \\ &= \left[\frac{t^3}{16}\right]_{-2}^{x} \\ &= \frac{x^3}{16} + \frac{1}{2}, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < -2, \\ \dfrac{x^3}{16} + \dfrac{1}{2}, & -2 \le x < 2, \\ 1, & 2 \le x < \infty. \end{cases}$$

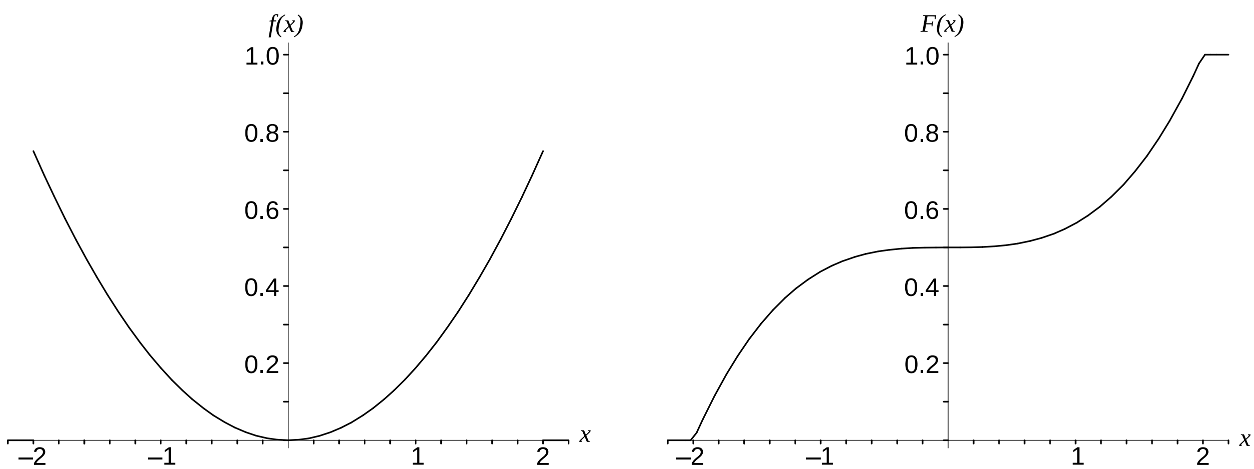


Figure 3.2–2: **(b)** Continuous distribution p.d.f. and c.d.f.

(c) **(i)**

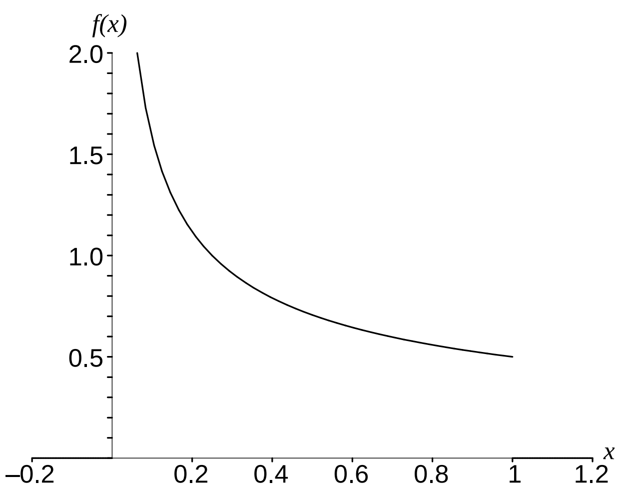
$$\begin{aligned} \int_0^1 \frac{c}{\sqrt{x}}\,dx &= 1 \\ 2c &= 1 \\ c &= 1/2. \end{aligned}$$

The p.d.f. in part (c) is unbounded.

(ii)

$$\begin{aligned} F(x) &= \textstyle\int_{-\infty}^{x} f(t)\,dt \\ &= \int_0^x \frac{1}{2\sqrt{t}}\,dt \\ &= \left[\sqrt{t}\right]_0^x = \sqrt{x}, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ \sqrt{x}, & 0 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$



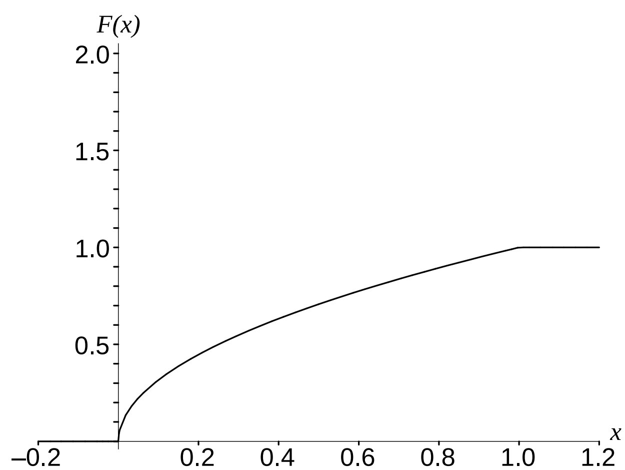


Figure 3.2–2: **(c)** Continuous distribution p.d.f. and c.d.f.

3.2–4 (a)

$$\begin{aligned}
\mu = E(X) &= \int_0^2 \frac{x^4}{4}\,dx \\
&= \left[\frac{x^5}{20}\right]_0^2 = \frac{32}{20} = \frac{8}{5}, \\
\sigma^2 = \mathrm{Var}(X) &= \int_0^2 \left(x - \frac{8}{5}\right)^2 \frac{x^3}{4}\,dx \\
&= \int_0^2 \left(\frac{x^5}{4} - \frac{4}{5}x^4 + \frac{16}{25}x^3\right) dx \\
&= \left[\frac{x^6}{24} - \frac{4x^5}{25} + \frac{4x^4}{25}\right]_0^2 \\
&= \frac{64}{24} - \frac{128}{25} + \frac{64}{25} \\
&\approx 0.1067, \\
\sigma &= \sqrt{0.1067} = 0.3266;
\end{aligned}$$

(b)

$$\begin{aligned}
\mu = E(X) &= \int_{-2}^2 \left(\frac{3}{16}\right) x^3\,dx \\
&= \left[\frac{3}{64}x^4\right]_{-2}^2 \\
&= \frac{48}{64} - \frac{48}{64} = 0, \\
\sigma^2 = \mathrm{Var}(X) &= \int_{-2}^2 \left(\frac{3}{16}\right) x^4\,dx \\
&= \left[\frac{3}{80}x^5\right]_{-2}^2 \\
&= \frac{96}{80} + \frac{96}{80} \\
&= \frac{12}{5}, \\
\sigma = \sqrt{\frac{12}{5}} &\approx 1.5492;
\end{aligned}$$

(c)
$$\begin{aligned}\mu = E(X) &= \int_0^1 \frac{x}{2\sqrt{x}}\,dx \\ &= \int_0^1 \frac{\sqrt{x}}{2}\,dx \\ &= \left[\frac{x^{3/2}}{3}\right]_0^1 = \frac{1}{3}, \\ \sigma^2 = \mathrm{Var}(X) &= \int_0^1 \left(x - \frac{1}{3}\right)^2 \frac{1}{2\sqrt{x}}\,dx \\ &= \int_0^1 \left(\frac{1}{2}x^{3/2} - \frac{2}{6}x^{1/2} + \frac{1}{18}x^{-1/2}\right)dx \\ &= \left[\frac{1}{5}x^{5/2} - \frac{2}{9}x^{3/2} + \frac{1}{9}x^{1/2}\right]_0^1 \\ &= \frac{4}{45}, \\ \sigma = \frac{2}{\sqrt{45}} &\approx 0.2981.\end{aligned}$$

3.2–6 **(a)**
$$\begin{aligned}M(t) &= \textstyle\int_0^\infty e^{tx}(1/2)x^2e^{-x}\,dx \\ &= \left[-\frac{x^2e^{-x(1-t)}}{2(1-t)} - \frac{xe^{-x(1-t)}}{(1-t)^2} - \frac{e^{-x(1-t)}}{(1-t)^3}\right]_0^\infty \\ &= \frac{1}{(1-t)^3}, \qquad t < 1;\end{aligned}$$

(b)
$$\begin{aligned}M'(t) &= \frac{3}{(1-t)^4} \\ M''(t) &= \frac{12}{(1-t)^5} \\ \mu &= M'(0) = 3 \\ \sigma^2 &= M''(0) - \mu^2 = 12 - 9 = 3.\end{aligned}$$

3.2–8 **(a)**
$$\begin{aligned}\int_1^\infty \frac{c}{x^2}\,dx &= 1 \\ \left[\frac{-c}{x}\right]_1^\infty &= 1 \\ c &= 1;\end{aligned}$$

(b) $E(X) = \displaystyle\int_1^\infty \frac{x}{x^2}\,dx = [\ln x]_1^\infty$, which is unbounded.

3.2–10 **(a)**

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x^3+1)/2, & -1 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

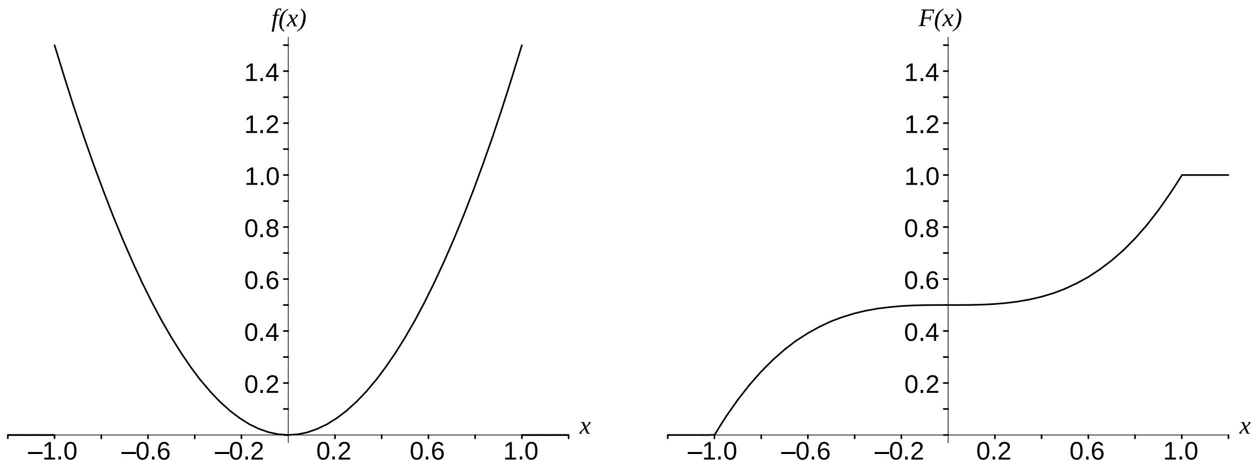


Figure 3.2–10: **(a)** $f(x) = (3/2)x^2$ and $F(x) = (x^3+1)/2$

(b)

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x+1)/2, & -1 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

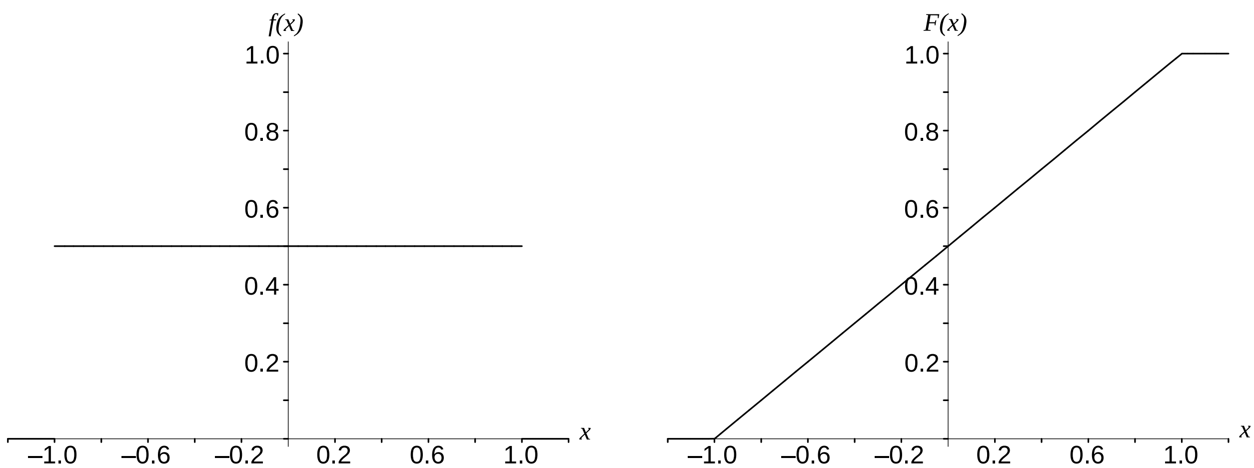


Figure 3.2–10: **(b)** $f(x) = 1/2$ and $F(x) = (x+1)/2$

(c)

$$F(x) = \begin{cases} 0, & -\infty < x < -1, \\ (x+1)^2/2, & -1 \le x < 0, \\ 1 - (1-x)^2/2, & 0 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$

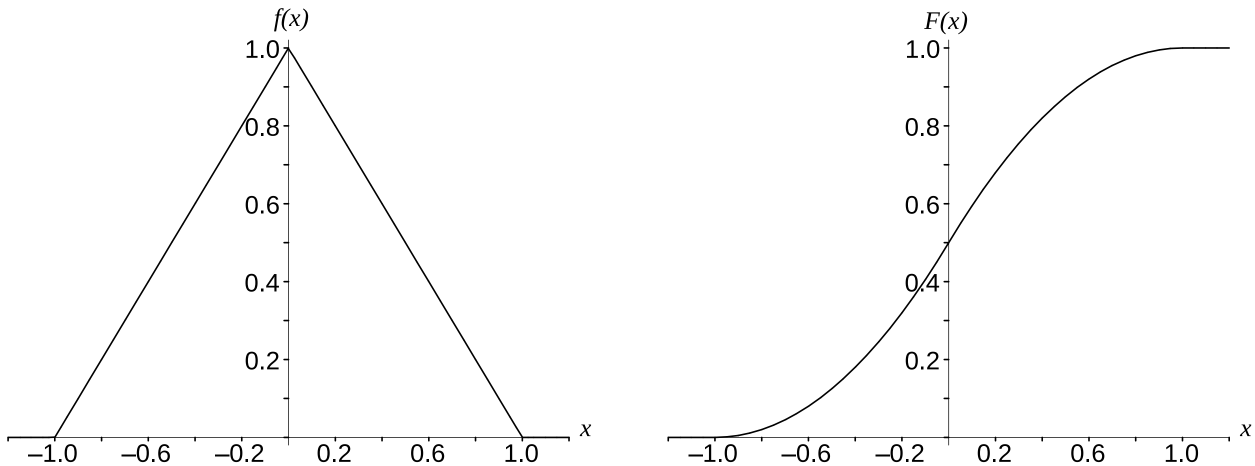


Figure 3.2–10: **(c)** $f(x)$ and $F(x)$ for Exercise 3.2-10(c)

3.2–12 **(a)** $R'(t) = \dfrac{M'(t)}{M(t)}$; $R'(0) = \dfrac{M'(0)}{M(0)} = M'(0) = \mu$;

(b)
$$\begin{aligned} R''(t) &= \frac{M(t)M''(t) - [M'(t)]^2}{[M(t)]^2}, \\ R''(0) &= M''(0) - [M'(0)]^2 = \sigma^2. \end{aligned}$$

3.2–14
$$\begin{aligned} M(t) &= \int_0^\infty e^{tx}(1/10)e^{-x/10}\,dx = \int_0^\infty (1/10)e^{-(x/10)(1-10t)}\,dx \\ &= (1-10t)^{-1}, \qquad t < 1/10. \\ R(t) &= \ln M(t) = -\ln(1-10t); \\ R'(t) &= 10/(1-10t) = 10(1-10t)^{-1}; \\ R''(t) &= 100(1-10t)^{-2}. \end{aligned}$$

Thus $\mu = R'(0) = 10$; $\sigma^2 = R''(0) = 100$.

3.2–16 **(b)**

$$F(x) = \begin{cases} 0, & -\infty < x \le 0, \\ \dfrac{x}{2}, & 0 < x \le 1, \\ \dfrac{1}{2}, & 1 < x \le 2, \\ \dfrac{x}{2} - \dfrac{1}{2} & 2 \le x < 3, \\ 1, & 3 \le x < \infty; \end{cases}$$

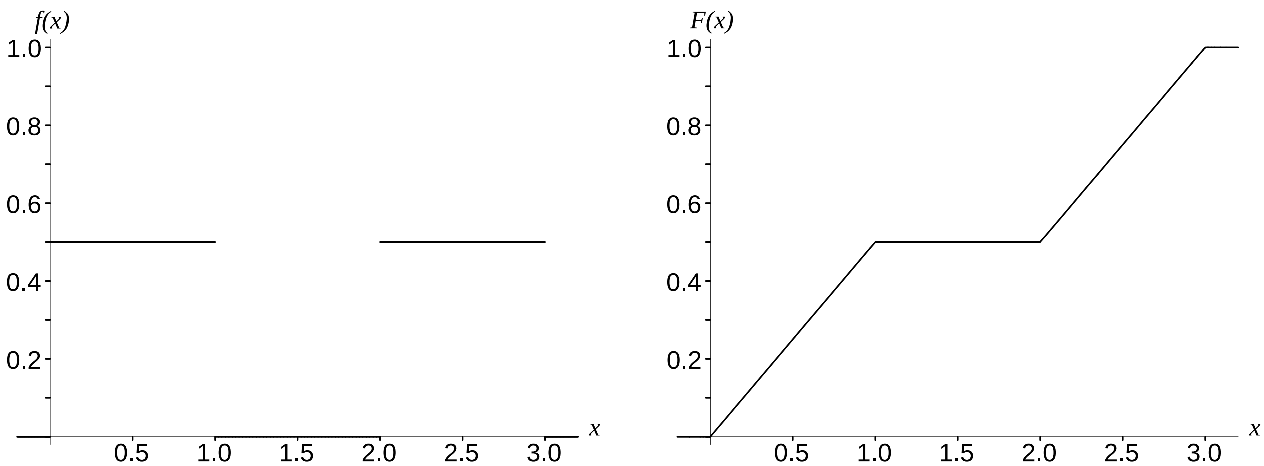


Figure 3.2–16: $f(x)$ and $F(x)$ for Exercise 3.2-16(a)

(c)
$$\begin{aligned} \frac{q_1}{2} &= 0.25 \\ q_1 &= 0.5, \end{aligned}$$

(d) $1 \le m \le 2$,

(e)
$$\begin{aligned} \frac{q_3}{2} - \frac{1}{2} &= 0.75 \\ \frac{q_3}{2} &= \frac{5}{4} \\ q_3 &= \frac{5}{2}. \end{aligned}$$

3.2–18 $F(x) = (x+1)^2/4, \qquad -1 < x < 1.$

(a)
$$\begin{aligned} F(\pi_{0.64}) = (\pi_{0.64}+1)^2/4 &= 0.64 \\ \pi_{0.64}+1 &= \sqrt{2.56} \\ \pi_{0.64} &= 0.6; \end{aligned}$$

(b)
$$\begin{aligned} (\pi_{0.25}+1)^2/4 &= 0.25 \\ \pi_{0.25}+1 &= \sqrt{1.00} \\ \pi_{0.25} &= 0; \end{aligned}$$

(c)
$$\begin{aligned} (\pi_{0.81}+1)^2/4 &= 0.81 \\ \pi_{0.81}+1 &= \sqrt{3.24} \\ \pi_{0.81} &= 0.8. \end{aligned}$$

3.2–20 **(a)** $\overline{x} = 1.3134$;

(b) $s = 0.5220$;

(c) The respective frequencies are 1, 6, 7, 6, 8, 10, 9, 13, 20, 20.

(d)

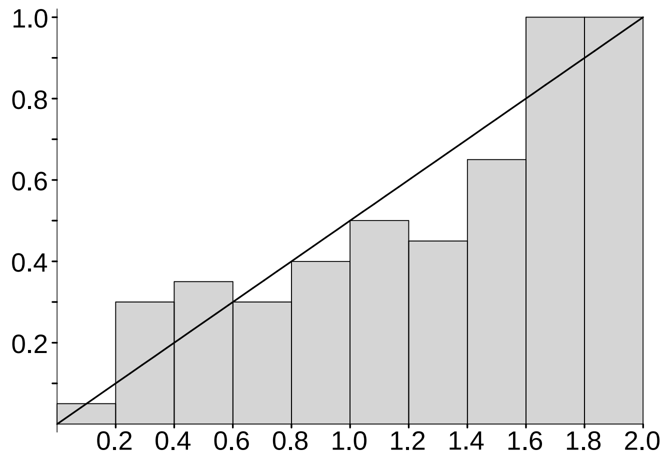


Figure 3.2–20: Relative frequency histogram with $f(x) = x/2$ superimposed

(e) $\mu = \displaystyle\int_0^2 x\left(\frac{x}{2}\right)dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}$;

$$\begin{aligned}\sigma^2 &= \int_0^2 x^2\left(\frac{x}{2}\right)dx - \left(\frac{4}{3}\right)^2 \\ &= \left[\frac{x^4}{8}\right]_0^2 - \frac{16}{9} = \frac{2}{9}.\end{aligned}$$

3.2–22 $P(X > 2) = \int_2^\infty 4x^3 e^{-x^4}dx = \left[-e^{-x^4}\right]_2^\infty = e^{-16}$.

3.2–24 **(a)** $P(X > 2000) = \int_{2000}^\infty (2x/1000^2)e^{-(x/1000)^2}dx = \left[-e^{-(x/1000)^2}\right]_{2000}^\infty = e^{-4}$;

(b)
$$\begin{aligned}\left[-e^{-(x/1000)^2}\right]_{\pi_{0.75}}^\infty &= 0.25 \\ e^{-(\pi_{0.75}/1000)^2} &= 0.25 \\ -(\pi_{0.75}/1000)^2 &= \ln(0.25) \\ \pi_{0.75} &= 1177.41;\end{aligned}$$

(c) $\pi_{0.10} = 324.59$;

(d) $\pi_{0.60} = 957.23$.

3.3 The Uniform and Exponential Distributions

3.3–2 $\mu = 0$, $\sigma^2 = 1/3$. See the figures for Exercise 3.2-10(b).

3.3–4 X is $U(4,5)$;

(a) $\mu = 9/2$; **(b)** $\sigma^2 = 1/12$; (c) 0.5.

3.3–6 (a)
$$\begin{aligned} P(10 < X < 30) &= \int_{10}^{30} \left(\frac{1}{20}\right) e^{-x/20}\,dx \\ &= \left[-e^{-x/20}\right]_{10}^{30} = e^{-1/2} - e^{-3/2}; \end{aligned}$$

(b)
$$\begin{aligned} P(X > 30) &= \int_{30}^{\infty} \frac{1}{20} e^{-x/20}\,dx \\ &= \left[-e^{-x/20}\right]_{30}^{\infty} = e^{-3/2}; \end{aligned}$$

(c)
$$\begin{aligned} P(X > 40 \mid X > 10) &= \frac{P(X > 40)}{P(X > 10)} \\ &= \frac{e^{-2}}{e^{-1/2}} = e^{-3/2}; \end{aligned}$$

(d) $\sigma^2 = \theta^2 = 400, \quad M(t) = (1 - 20t)^{-1}$.

(e)
$$\begin{aligned} P(10 < X < 30) &= 0.383, \text{ close to the relative frequency } \frac{35}{100}, \\ P(X > 30) &= 0.223, \text{ close to the relative frequency } \frac{23}{100}, \\ P(X > 40 \mid X > 10) &= 0.223, \text{ close to the relative frequency } \frac{14}{58} = 0.241. \end{aligned}$$

3.3–8 (a) $f(x) = \left(\frac{2}{3}\right) e^{-2x/3}, \qquad 0 \le x < \infty;$

(b) $P(X > 2) = \displaystyle\int_2^{\infty} \frac{2}{3} e^{-2x/3}\,dx = \left[-e^{-2x/3}\right]_2^{\infty} = e^{-4/3}.$

3.3–10 (a) $P(W \ge 6) = \displaystyle\int_6^{\infty} \frac{1}{3} e^{-x/3}\,dx = \left[-e^{-x/3}\right]_6^{\infty} = e^{-2};$

(b) $P(W > 12 \mid W > 6) = \dfrac{P(W > 12)}{P(W > 6)} = \dfrac{e^{-4}}{e^{-2}} = e^{-2}.$

3.3–12 Let $F(x) = P(X \le x)$. Then
$$\begin{aligned} P(X > x + y \mid X > x) &= P(X > y) \\ \frac{1 - F(x+y)}{1 - F(x)} &= 1 - F(y). \end{aligned}$$
That is, with $g(x) = 1 - F(x)$, $g(x+y) = g(x)g(y)$. This functional equation implies that
$$1 - F(x) = g(x) = a^{cx} = e^{(cx)\ln a} = e^{bx}$$
where $b = c\ln a$. That is, $F(x) = 1 - e^{bx}$. Since $F(\infty) = 1$, b must be negative, say $b = -\lambda$ with $\lambda > 0$. Thus $F(x) = 1 - e^{-\lambda x}$, $0 \le x$, the distribution function of an exponential distribution.

3.3–14
$$\begin{aligned} E[v(T)] &= \textstyle\int_0^3 100(2^{3-t}-1)e^{-t/5}/5dt \\ &= \textstyle\int_0^3 -20e^{-t/5}dt + 100\int_0^3 e^{(3-t)\ln 2}e^{-t/5}/5dt \\ &= -100(1-e^{-0.6}) + 100e^{3\ln 2}\textstyle\int_0^3 e^{-t\ln 2}e^{-t/5}/5dt \\ &= -100(1-e^{-0.6}) + 100e^{3\ln 2}\left[-\frac{e^{-(\ln 2+0.2)t}}{\ln 2+0.2}\right]_0^3 \\ &= 121.734. \end{aligned}$$

3.3–16
$$\begin{aligned} E(\text{profit}) &= \textstyle\int_0^n [x-0.5(n-x)]\frac{1}{200}\,dx + \int_n^{200}[n-5(x-n)]\frac{1}{200}\,dx \\ &= \frac{1}{200}\left[\frac{x^2}{2}+\frac{(n-x)^2}{4}\right]_0^n + \frac{1}{200}\left[6nx-\frac{5x^2}{2}\right]_n^{200} \\ &= \tfrac{1}{200}[-3.25n^2+1200n-100000] \\ \text{derivative} &= \tfrac{1}{200}\,[-6.5n+1200] = 0 \\ n &= \frac{1200}{6.5} \approx 185. \end{aligned}$$

3.3–18 **(a)**
$$\begin{aligned} P(X>40) &= \int_{40}^{\infty}\frac{3}{100}e^{-3x/100}\,dx \\ &= \left[-e^{-3x/100}\right]_{40}^{\infty} = e^{-1.2}; \end{aligned}$$

(b) Flaws occur randomly so we are observing a Poisson process.

3.3–20
$$\begin{aligned} F(x) &= \int_{-\infty}^{x}\frac{e^{-w}}{(1+e^{-w})^2}\,dw = \frac{1}{1+e^{-x}}, \qquad -\infty<x<\infty. \\ G(y) &= P\left[\frac{1}{1+e^{-X}}\le y\right] = P\left[X\le -\ln\left(\frac{1}{y}-1\right)\right] \\ &= \frac{1}{1+\left(\frac{1}{y}-1\right)} = y, \qquad 0<y<1, \end{aligned}$$

the $U(0,1)$ distribution function.

3.4 The Gamma and Chi-Square Distributions

3.4–2 Either use integration by parts or

$$\begin{aligned} F(x) &= P(X\le x) \\ &= 1-\sum_{k=0}^{\alpha-1}\frac{(\lambda x)^k e^{-\lambda x}}{k!}. \end{aligned}$$

Thus, with $\lambda = 1/\theta = 1/4$ and $\alpha = 2$,

$$\begin{aligned} P(X<5) &= 1-e^{-5/4}-\left(\frac{5}{4}\right)e^{-5/4} \\ &= 0.35536. \end{aligned}$$

3.4–4 The moment generating function of X is $M(t) = (1-\theta t)^{-\alpha},\ t < 1/\theta$. Thus

$$\begin{aligned} M'(t) &= \alpha\theta(1-\theta t)^{-\alpha-1} \\ M''(t) &= \alpha(\alpha+1)\theta^2(1-\theta t)^{-\alpha-2}. \end{aligned}$$

The mean and variance are

$$\begin{aligned} \mu &= M'(0) = \alpha\theta \\ \sigma^2 &= M''(0) - (\alpha\theta)^2 = \alpha(\alpha+1)\theta^2 - (\alpha\theta)^2 \\ &= \alpha\theta^2. \end{aligned}$$

3.4–6 (See Figure 8.10-2, page 579, in the textbook.)

(a) $$\begin{aligned} f(x) &= \frac{14.7^{100}}{\Gamma(100)}\, x^{99} e^{-14.7x}, \qquad 0 \le x < \infty, \\ \mu &= 100(1/14.7) = 6.80, \quad \sigma^2 = 100(1/14.7)^2 = 0.4628; \end{aligned}$$

(b) $\overline{x} = 6.74,\ s^2 = 0.4617$;

(c) $9/25 = 0.36$.

3.4–8 **(a)** W has a gamma distribution with $\alpha = 7,\ \theta = 1/16$.

(b) Using Table III in the Appendix,

$$\begin{aligned} P(W \le 0.5) &= 1 - \sum_{k=0}^{6} \frac{8^k e^{-8}}{k!} \\ &= 1 - 0.313 = 0.687, \end{aligned}$$

because here $\lambda w = (16)(0.5) = 8$.

3.4–10 $a = 5.226, \quad b = 21.03$.

3.4–12 Since the m.g.f. is that of $\chi^2(24)$, we have **(a)** $\mu = 24$; **(b)** $\sigma^2 = 48$; and **(c)** 0.89, using Table IV.

3.4–14 Note that $\lambda = 5/10 = 1/2$ is the mean number of arrivals per minute. Thus $\theta = 2$ and the p.d.f. of the waiting time before the eighth toll is

$$\begin{aligned} f(x) &= \frac{1}{\Gamma(8)2^8}\, x^{8-1} e^{-x/2} \\ &= \frac{1}{\Gamma\left(\frac{16}{2}\right) 2^{16/2}}\, x^{16/2-1} e^{-x/2}, \qquad 0 < x < \infty, \end{aligned}$$

the p.d.f. of a chi-square distribution with $r = 16$ degrees of freedom. Using Table IV,

$$P(X > 26.30) = 0.05.$$

3.4–16 $P(X > 30.14) = 0.05$ where X denotes a single observation. Let W equal the number out of 10 observations that exceed 30.14. Then the distribution of W is $b(10,\ 0.05)$. Thus

$$P(W = 2) = 0.9885 - 0.9139 = 0.0746.$$

3.5 Distributions of Functions of a Random Variable

3.5–2 Here $x = \sqrt{y}$, $D_y(x) = 1/2\sqrt{y}$ and $0 < x < \infty$ maps onto $0 < y < \infty$. Thus

$$g(y) = \sqrt{y}\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2}e^{-y/2}, \qquad 0 < y < \infty.$$

3.5–4 **(a)**

$$F(x) = \begin{cases} 0, & x < 0, \\ \displaystyle\int_0^x 2t\,dt = x^2, & 0 \le x < 1, \\ 1, & 1 \le x, \end{cases}$$

(b) Let $y = x^2$; so $x = \sqrt{y}$. Let Y be $U(0,1)$; then $X = \sqrt{Y}$ has the given x-distribution.

(c) Repeat the procedure outlined in part (b) 10 times.

(d) Order the 10 values of x found in part (c), say $x_1 < x_2 < \cdots < x_{10}$ and plot the 10 points $(x_i, \sqrt{i/11})$, $i = 1, 2, \ldots, 10$, where $11 = n + 1$.

3.5–6 It is easier to note that

$$\frac{dy}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{(1+e^{-x})^2}{e^{-x}}.$$

Say the solution of x in terms of y is given by x^*. Then the p.d.f. of Y is

$$g(y) = \frac{e^{-x^*}}{(1+e^{-x^*})^2}\left|\frac{(1+e^{-x^*})^2}{e^{-x^*}}\right| = 1, \qquad 0 < y < 1,$$

as $-\infty < x < \infty$ maps onto $0 < y < 1$. Thus Y is $U(0,1)$.

3.5–8

$$\begin{aligned} x &= \left(\frac{y}{5}\right)^{10/7} \\ \frac{dx}{dy} &= \frac{10}{7}\left(\frac{y}{5}\right)^{3/7}\left(\frac{1}{5}\right) \\ f(x) &= e^{-x}, \qquad 0 < x < \infty \\ g(y) &= e^{-(y/5)^{10/7}}\left(\frac{2}{7}\right)\left(\frac{1}{5}\right)^{3/7} y^{3/7} \\ &= \frac{10/7}{5^{10/7}}y^{3/7}e^{-(y/5)^{10/7}}, \qquad 0 < y < \infty. \end{aligned}$$

(The reason for writing the p.d.f. in that form is because Y has a Weibull distribution with $\alpha = 10/7$ and $\beta = 5$. See page 184 in the textbook.)

3.5–10 Since $-1 < x < 3$, we have $0 \le y < 9$.

When $0 < y < 1$, then

$$x_1 = -\sqrt{y}, \quad \frac{dx_1}{dy} = \frac{-1}{2\sqrt{y}}; \qquad x_2 = \sqrt{y}, \quad \frac{dx_2}{dy} = \frac{1}{2\sqrt{y}}.$$

When $1 < y < 9$, then

$$x = \sqrt{y}, \quad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}.$$

Thus

$$g(y) = \begin{cases} \dfrac{1}{4}\cdot\left|\dfrac{-1}{2\sqrt{y}}\right| + \dfrac{1}{4}\cdot\left|\dfrac{1}{2\sqrt{y}}\right| = \dfrac{1}{4\sqrt{y}} & 0 < y < 1, \\ \dfrac{1}{4}\cdot\left|\dfrac{1}{2\sqrt{y}}\right| = \dfrac{1}{8\sqrt{y}} & 1 \le y < 9. \end{cases}$$

3.5–12

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\,dx \\ &= \lim_{a\to-\infty}\left[\frac{1}{2\pi}\ln(1+x^2)\right]_a^0 + \lim_{b\to+\infty}\left[\frac{1}{2\pi}\ln(1+x^2)\right]_0^b \\ &= \frac{1}{2\pi}\left[\lim_{a\to-\infty}\{-\ln(1+a^2)\} + \lim_{b\to+\infty}\ln(1+b^2)\right]. \end{aligned}$$

$E(X)$ does not exist because neither of these limits exists.

3.5–14 Simulate observations of the Cauchy random variable X using

$$y = \int_{-\infty}^{x} \frac{1}{\pi(1+w^2)}\,dw$$

or, equivalently,

$$x = \tan[\pi(y - 1/2)],$$

where y is an observation from the $U(0,1)$ distribution.

3.6 Additional Models

3.6–2 With $b = \ln 1.1$,

$$\begin{aligned} G(w) &= 1 - \exp\left[-\frac{a}{\ln 1.1}e^{w\ln 1.1} + \frac{a}{\ln 1.1}\right] \\ G(64) - G(63) &= 0.01 \\ a &= 0.00002646 = \frac{1}{37792.19477} \\ P(W \le 71 \,|\, 70 < W) &= \frac{P(70 < W \le 71)}{P(70 < W)} \\ &= 0.0217. \end{aligned}$$

3.6–4

$$\begin{aligned} \lambda(w) &= ae^{bw} + c \\ H(w) &= \textstyle\int_0^w (ae^{bt} + c)\,dt \\ &= \frac{a}{b}\left(e^{bw} - 1\right) + cw \\ G(w) &= 1 - \exp\left[-\frac{a}{b}\left(e^{bw} - 1\right) - cw\right], \qquad 0 < \infty \\ g(w) &= (ae^{bw} + c)e^{-\frac{a}{b}(e^{bw}-1) - cw}, \qquad 0 < \infty. \end{aligned}$$

3.6–6 **(a)** $1/4 - 1/8 = 1/8$; **(b)** $1/4 - 1/4 = 0$;
(c) $3/4 - 1/4 = 1/2$; **(d)** $1 - 1/2 = 1/2$;
(e) $3/4 - 3/4 = 0$; **(f)** $1 - 3/4 = 1/4$.

3.6–8 There is a discrete point of probability at $x = 0$, $P(X = 0) = 1/3$, and $F'(x) = (2/3)e^{-x}$ for $0 < x$. Thus

$$\begin{aligned}\mu = E(X) &= (0)(1/3) + \int_0^\infty x(2/3)e^{-x}dx \\ &= (2/3)[-xe^{-x} + e^{-x}]_0^\infty = 2/3, \\ E(X^2) &= (0)^2(1/3) + \int_0^\infty x^2(2/3)e^{-x}dx \\ &= (2/3)[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}]_0^\infty = 4/3,\end{aligned}$$

so

$$\sigma^2 = \text{Var}(X) = 4/3 - (2/3)^2 = 8/9.$$

3.6–10 For the uncensored distribution,

$$F'(x) = 3(10^3)(10 + x)^{-4}, \qquad 0 < x < \infty.$$

Thus

$$\begin{aligned}E(X) &= \int_0^\infty x(3000)(10 + x)^{-4}\,dx \\ &= \left[-1000x(10 + x)^{-3} - 500(10 + x)^{-2}\right]_0^\infty = 5.\end{aligned}$$

For the censored distribution,

$$\begin{aligned}E(Y) &= \int_0^{10} y(3000)(10 + y)^{-4}\,dy + 10(1 - [1 - 1/8]) \\ &= \left[-1000y(10 + y)^{-3} - 500(10 + y)^{-2}\right]_0^{10} + 10(1/8) \\ &= -\frac{10,000}{8000} - \frac{500}{400} + \frac{500}{100} + \frac{10}{8} = 3.75.\end{aligned}$$

3.6–12
$$T = \begin{cases} X, & X \le 4, \\ 4, & 4 < X; \end{cases}$$

$$\begin{aligned}E(T) &= \int_0^4 x\left(\frac{1}{5}\right)e^{-x/5}\,dx + \int_4^\infty 4\left(\frac{1}{5}\right)e^{-x/5}\,dx \\ &= \left[-xe^{-x/5} - 5e^{-x/5}\right]_0^4 + 4\left[-e^{-x/5}\right]_4^\infty \\ &= 5 - 4e^{-4/5} - 5e^{-4/5} + 4e^{-4/5} \\ &= 5 - 5e^{-4/5} \approx 2.753.\end{aligned}$$

3.6–14 (a)

$$\begin{aligned} t &= \ln x \\ x &= e^t \\ \frac{dx}{dt} &= e^t \\ g(t) &= f(e^t)\frac{dx}{dt} = e^t e^{-e^t}, \qquad -\infty < t < \infty. \end{aligned}$$

(b)

$$\begin{aligned} t &= \alpha + \beta \ln w \\ \frac{dt}{dw} &= \frac{\beta}{w} \\ h(w) &= e^{\alpha+\beta \ln w} e^{-e^{\alpha+\beta \ln w}} \left(\frac{\beta}{w}\right) \\ &= \beta w^{\beta-1} e^{\alpha} e^{-w^{\beta} e^{\alpha}}, \qquad 0 < w < \infty. \end{aligned}$$